

# A note on generic types

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## Abstract

In a stable abelian group, we characterize generic types of cosets of type-definable subgroups.

The following remark is part of the folklore. It was stated by the author 1990 in a letter to Daniel Lascar.

**Theorem 1.** *Let  $G$  be a stable abelian group.  $a, b$  and  $c$  pairwise independent over  $0$  and  $a + b + c = 0$ . Then:*

1. *The strong types of  $a, b$  and  $c$  all have the same connected stabilizer  $U$ .*
2.  *$a, b$ , and  $c$  are generic elements of  $\text{acl}^{\text{eq}}(0)$ -definable cosets of  $U$ .*

If  $G$  is totally-transcendent, it follows that  $a, b$  and  $c$  have the same Morley rank over  $0$ , namely  $\text{den rank}$  of  $U$ . Moreover  $U$  is definable.

Let  $p$  and  $q$  be strong types over  $0$ . Then

$$\text{Hom}(p, q) = \{g \in G \mid \forall a \ a \models p|_g \Rightarrow a + g \models q|_g\}$$

is an  $\text{acl}^{\text{eq}}(0)$ -type definable (if  $G$  is totally transcendent: definable) subset of  $G$ . Of course  $\text{Hom}(p, p) = \text{Stab}(p)$ .

**Lemma 2.** *Let  $p, q$  and  $r$  be strong types. Then*

- *For all  $A \subset G$  and  $g \in \text{Hom}(p, q)$  we have*

$$a \models p|_{g,A} \Rightarrow a + g \models q|_{g,A}.$$

- $\text{Hom}(p, q) + \text{Hom}(q, r) \subset \text{Hom}(p, r)$
- $\text{Hom}(p, q) = -\text{Hom}(q, p) = \text{Hom}(-q, -p)$
- $0 \in \text{Hom}(p, p)$
- *If  $\text{Hom}(p, q)$  is non-empty and  $G$  totally-transcendent,*

$$\text{MR}(\text{Hom}(p, q)) \leq \text{MR}(p) = \text{MR}(q).$$

*Proof.* Let  $a$  be a realization of  $p$ , which is independent of  $g, A$ . Then  $a \downarrow_g A$  implies that  $a + g \downarrow_g A$ . Since  $a + g \downarrow g$ , we have  $a + g \downarrow g, A$ .

Let  $g \in \text{Hom}(p, q)$ ,  $h \in \text{Hom}(q, r)$  and  $a$  be a realization of  $p$  which is independent of  $g + h$ . We may assume that  $a$  is independent of  $g, h$ . Then  $a + g$  is a realization of  $q$ , which is also independent of  $g, h$ . Therefore  $a + g + h$  is a realization of  $r$ , which is independent of  $g, h$  and therefore of  $g + h$ .

Let  $g \in \text{Hom}(p, q)$  and  $a$  be a realization of  $p$ , which is independent of  $g$ . Then is

$$\text{MR}(a) = \text{MR}(a/g) = \text{MR}(a + g/g) = \text{MR}(a + g).$$

This implies  $\text{MR}(p) = \text{MR}(q)$ . From

$$\text{MR}(g) = \text{MR}(g/a) = \text{MR}(a + g/a) \leq \text{MR}(a + g)$$

follows  $\text{MR}(\text{Hom}(p, q)) \leq \text{MR}(q)$ .  $\square$

Assume now that  $a, b$  and  $c$  are as in the theorem,  $p, q$  and  $r$  the strong types of  $a, b$  and  $c$ . Then, trivially,

- $p(G) \subset \text{Hom}(q, -r) = \text{Hom}(r, -q)$
- $q(G) \subset \text{Hom}(r, -p) = \text{Hom}(p, -r)$
- $r(G) \subset \text{Hom}(p, -q) = \text{Hom}(q, -p)$ .

It follows

- $p(G) - p(G) \subset \text{Stab}(q), \text{Stab}(r)$
- $q(G) - q(G) \subset \text{Stab}(r), \text{Stab}(p)$
- $r(G) - r(G) \subset \text{Stab}(p), \text{Stab}(q)$ .

Since on the other hand

- $\text{Stab}(p) \subset p(G) - p(G)$
- $\text{Stab}(q) \subset q(G) - q(G)$
- $\text{Stab}(r) \subset r(G) - r(G)$ ,

all stabilizers are equal to

$$U = p(G) - p(G) = q(G) - q(G) = r(G) - r(G).$$

Therefore  $p(G), q(G), r(G)$  all lie in  $\text{acl}^{\text{eq}}(0)$ -definable cosets of  $U$ . Since  $U$  is the stabilizer of each of these types,  $p, q, r$  are generic types and  $U$  is connected. This proves the theorem.